

A FAMILY OF MEASURES ON SYMMETRIC GROUPS AND THE FIELD WITH ONE ELEMENT

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ABSTRACT. For each $n \geq 1$ this paper considers a one-parameter family of complex-valued measures on the symmetric group S_n , depending on a complex parameter z . For parameter values $z = q = p^f$ this measure describes splitting probabilities of monic degree n polynomials over $\mathbb{F}_q[X]$, conditioned on being squarefree. It studies these measures in the case $z = 1$, and shows that they have an interesting internal structure having a representation theoretic interpretation. These measures may encode data relevant to the hypothetical “field with one element \mathbb{F}_1 .”

1. INTRODUCTION

This paper considers a one-parameter family of complex-valued measures on the symmetric group S_n , called *z-splitting measures*, with complex parameter z , studied by the author and B. L. Weiss in [13]. Each measure is constant on conjugacy classes C_λ of S_n , which are indexed by partitions λ labeling the (common) cycle structure of all elements $g \in S_n$ in the conjugacy class. To define them, for each degree $m \geq 1$ we first define the *m-th necklace polynomial* $M_m(X)$ by

$$M_m(X) := \frac{1}{m} \sum_{d|m} \mu(d) X^{m/d}.$$

where $\mu(d)$ is the Möbius function. The *z-splitting measure* $\nu_{n,z}$ is defined on conjugacy classes C_λ of S_n by

$$\nu_{n,z}^*(C_\lambda) := \frac{1}{z^{n-1}(z-1)} \prod_{j=1}^n \binom{M_j(z)}{m_j(\lambda)}. \quad (1.1)$$

in which $m_j = m_j(\lambda)$ counts the number of cycles in $g \in S_n$ of length j , and for a complex number z we interpret $\binom{z}{k} := \frac{(z)_k}{k!} = \frac{z(z-1)\cdots(z-k+1)}{k!}$.

These measures were termed *z-splitting measures* in [13] because for parameter values $z = q = p^f$, a prime power, they described (normalized) splitting probabilities of polynomials over finite fields \mathbb{F}_q , restricted to squarefree polynomials. In [13] it was shown that for all integers $k \neq 0, 1$ the measures at $z = k$, $\nu_{n,k}$, are probability measures. They also define a probability measure at $z = \infty$, which is the uniform measure on S_n .

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The object of this paper is to study in detail this measure at $z = 1$, which is the remaining integer value where the z -splitting measure is well-defined, as we observe in Lemma 2.5 (the formulas diverge at $z = 0$). We call $\nu_{n,1}^*$ the 1-splitting measure. We show that the 1-splitting measure is a signed measure for $n \geq 3$; the failure of nonnegativity to hold at the value $z = 1$ is a distinctive special feature of this value.

This paper studies these measures phenomenologically and shows they have an internal structure which can be interpreted in terms of the representation theory of S_n . At the same time this internal structure respects the multiplicative structure of integers.

1.1. Main Results. We show that the 1-splitting distributions have the following properties.

- (1) The 1-splitting measure $\nu_{n,1}^*$ is supported on the conjugacy classes of S_n whose associated partitions are rectangles $[b^a]$ with $ab = n$ or else are rectangles plus a single extra box, those of type $[d^c, 1]$ with $cd = n - 1$ (Theorem 3.1). These are exactly the *Springer regular elements* of the Coxeter group S_n , in the sense of [25], [20].
- (2) The signed measure $\nu_{n,1}^*$ can be uniquely written as a sum of two (signed) measures

$$\nu_{n,1}^* = \omega_n + \omega_{n-1}^*,$$

in which the measure ω_n is supported on partitions of type $[b^a]$, and ω_{n-1}^* is supported on partitions of type $[d^c, 1]$, such that the value of ω_{n-1}^* summed over the conjugacy class $[d^c, 1]$ agrees with the measure ω_{n-1} on S_{n-1} summed over the conjugacy class $[d^c]$. Thus the family of measures $\{\nu_{n,1}^* : n \geq 1\}$ are in effect built up out of the family of measures $\{\omega_n : n \geq 1\}$.

The two signed measures ω_n and ω_{n-1}^* overlap on the identity conjugacy class $[1^n]$, and the (signed) mass there must be properly subdivided between the two measures (Theorem 3.1). The measures ω_n are computed explicitly (Theorem 3.2).

- (3) The measures ω_n respect the multiplicative structure of integers, in the following sense: If n has prime factorization

$$n = \prod_i p_i^{e_i}$$

and also factors as $n = ab$ then

$$\omega_n(C_{[b^a]}) = \prod_i \omega_{p_i^{e_i}}(C_{[(b_i)^{a_i}]}),$$

in which $b_i = p_i^{e_i,2}$ (and $a_i = p_i^{e_i,1}$) represent the maximal power of p_i dividing b (resp. a). Here the values $a = 1$ or $b = 1$ are permitted. In this factorization the $\omega_{p_i}(\cdot)$ values at primes $p_i \geq 3$ are always positive, and only the prime $p_i = 2$ contributes signed terms to the measure value (Theorem 3.3).

- (4) For odd $n = 2m + 1$ the measure ω_n is a positive measure of total mass 1. For even $n = 2m$ the measure ω_{2m} is a signed measure of total mass 0, and the absolute value measure $|\omega_{2m}|$ has total mass 1 (Theorem 3.3).
- (5) There is a probabilistic sampling construction of the genuine probability distributions $|\omega_n|$ for all $n \geq 2$ (Theorem 4.1). There is a probabilistic sampling construction (adding signs) for the signed distributions ω_{2m} for even integers $2m$ (Theorem 4.2).
- (6) The scaled measures $n!|\omega_n|$ and $n!\omega_n$ (these measures are identical for odd $n = 2m + 1$), take integer values on conjugacy classes. The class function $n!|\omega_n|$ is the character of a genuine representation. For $n = 2m$ the class function $-(2m)!\omega_{2m}$ is the character of a genuine representation. For the character $n!|\omega_n|$ the corresponding genuine representation is the representation induced from the trivial representation χ_{triv} on any cyclic subgroup of S_n that is generated by an n -cycle (Theorem 5.1). In the case that $n = 2m$ is even, the representation having character $-(2m)!\omega_{2m}$ is the genuine representation induced from the sign character representation χ_{sgn} on the cyclic group generated by an $2m$ -cycle (Theorem 5.2). Finally, the class function $(-1)^n n! \omega_{n-1}^*$ takes integer values and is the character of a genuine representation of S_n (Theorem 5.4).

1.2. Discussion. The value $z = 1$ can be viewed formally as corresponding to the hypothetical (i.e. nonexistent) “field with one element \mathbb{F}_1 ,” and the resulting signed measure can therefore formally be viewed as a statistic determining “splitting probabilities” for degree n polynomials over the field \mathbb{F}_1 with one element.

The concept of a (hypothetical) “field with one element \mathbb{F}_1 ” was suggested in 1957 by Tits [28] as a way to describe the uniformity of certain phenomena on finite geometries associated to algebraic groups coordinatized by points over a finite field \mathbb{F}_p . His theory of buildings related algebraic groups to certain simplicial complexes. For a Chevalley group scheme G he said that the Weyl group of G can be viewed as the group of points of G over the “field with one element.” More generally one may consider finite incidence geometries over finite fields \mathbb{F}_q (compare [4]) where there again may be interesting degenerate geometric objects associated to “the field with one element.”

Another direction of generalization associates to certain algebraic varieties defined over \mathbb{Q} (or \mathbb{Z}) arithmetic statistics obtained by counting points under reduction modulo p for varying primes p , e.g. counting points on the variety over \mathbb{F}_p . One can also evaluate more exotic statistics associated with vector bundles and cohomology for such varieties, viewed over finite fields \mathbb{F}_q . In restricted circumstances these statistics may have the feature of being interpolatable by a rational function $R(z)$ in a parameter z which for $z = q = p^k$ interpolates the statistics of the geometric object. In [13] we termed this property the *rational function interpolation property*. Whenever this property holds one may insert the value $z = 1$ and define the resulting value $R(1)$ to be the analogue statistic over “the field with one element \mathbb{F}_1 .”

The rational function interpolation property is known to hold for counting points over \mathbb{F}_q on nonsingular toric varieties defined over \mathbb{Q} , compare [15], [16]. However it is known that only a restricted class of varieties yield statistics having the rational function interpolation property. There has been much recent work on developing notions of generalized algebraic geometry “over \mathbb{F}_1 ”, of which we may mention [5], [6], [7], [8], [12], [14], [15], [16], [24], [27], [29].

The distributions on S_n in this paper arose in connection with splitting problems for polynomials defined over finite fields \mathbb{F}_q , as studied in [30] and [13]. They have the rational function interpolation property since (1.1) implies that for each fixed conjugacy class C_λ the functions $\nu_{n,z}^*(C_\lambda)$ are rational functions of the parameter z whose poles can only occur at $z = 0, 1, \infty$; in fact poles may only occur at $z = 0$. An interesting feature of this interpretation is that the probability of a random degree n polynomial over \mathbb{F}_q being square-free is exactly $1 - \frac{1}{q}$, independent of its degree n . Choosing $q = 1$ yields the limit probability 0 of being square-free. The results of this paper however concern a conditional probability, normalized to specify total mass 1 (not mass $(1 - \frac{1}{q})$), which has a nontrivial limit at $q = 1$. Perhaps this limiting process indicates that the measures we study should be viewed as attached to the “tangent space to the field of one element \mathbb{F}_1 ” rather than to the hypothetical field \mathbb{F}_1 .

We do not have a conceptual geometric explanation of this $q \rightarrow 1$ construction. The fact that the support of these measures is the Springer regular elements of S_n suggests it may have a geometric interpretation. The fact that the limit measure is signed suggests that the geometry will be different in some aspect from \mathbb{F}_1 -type statistics associated with counting points on closed varieties twisted by local systems; it seems to be associated to an open variety. A geometric interpretation would align the observations made here with various known geometric and topological “field of one element” constructions.

In terms of geometry, the z -splitting measure on S_n at $z = q$ is associated with properties of the \mathbb{F}_q -points of the open (noncompact) variety $Z_n := \mathbb{P}^n \setminus \{H_n, L_n\}$ where the projective space \mathbb{P}^n is identified with the coefficients (a_0, a_1, \dots, a_n) associated to the polynomial $f(X) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n$, with $L_n := \{a_0 = 0\}$ being the hyperplane “at infinity” and the *discriminant locus* $H_n := \{\text{Disc}(f(X)) = 0\}$ being the hypersurface cut out by the discriminant of $f(X)$, given by a homogeneous polynomial of degree n . The condition $\text{Disc}(f(X)) \neq 0$ specifies that $f(X)$ has a square-free factorization. (Removing L_n takes away monic polynomials of lower degrees). The group S_n acts on the roots of $f(X)$ and on the possible factorizations of $f(X)$, and the \mathbb{F}_q -points are distinguished by the S_n action. The S_n -action distinguishing factorizations is associated to a configuration space $\text{Conf}(n)$ of n distinct points (z_1, \dots, z_n) subject to the distinctness constraint $z_i \neq z_j$, with associated polynomial $f(X) = a_0 \prod_{i=1}^n (X - z_i)$, see Church, Ellenberg and Farb [1], [2]. Their work studies aspects of the S_n -action on homology of this variety and its numbers of \mathbb{F}_q -points. Their method extracts asymptotic information as $n \rightarrow \infty$ on various statistics on

the occurrence of certain families of representations. The limit $q = 1$ would represent a new sort of (unstable) limit for their statistics. It may be that the 1-splitting measure is encoding geometric information about the discriminant locus.

1.3. Plan of the Paper. Section 2 recalls basic facts about the z -splitting distributions following [13]. Section 3 describes properties of the 1-splitting measures. It splits them into a sum of two simpler measures ω_n and ω_{n-1}^* . Section 4 gives a probabilistic interpretation of ω_n and $|\omega_n|$. Section 5 gives a representation-theoretic interpretation of ω_n and ω_{n-1}^* . Section 6 makes some concluding remarks.

1.4. Notation. (1) p denotes a prime, and $q = p^f$ denotes a prime power.

(2) This paper follows the notation of Macdonald [17] for partitions, denoting them λ , and denotes cycle numbers $m_i(\lambda)$. This notation differs from [13], which used μ for partitions and cycle number $c_i(\mu) := |\{j : \mu_j \geq i\}|$ corresponds to Macdonald's $m_i(\lambda)$.

(3) For complex-valued functions f on a group G , either a measure or a character, given a subset $Y \subseteq G$, we let $f(Y) := \sum_{g \in Y} f(g)$. The absolute value function $|f| : G \rightarrow \mathbb{R}$ is defined by $|f|(g) := |f(g)|$.

2. PRELIMINARY FACTS

We review basic facts on splitting measures. To emphasize the rational function aspect, we sometimes use the alternate notation

$$R_\lambda(z) := \nu_{n,z}^*(C_\lambda),$$

and $R_g(z) := \nu_{n,z}^*(g)$ for $g \in S_n$. This notation facilitates discussion of these rational functions in neighborhoods of points in the z -sphere where they have poles. The splitting measures are constructed using necklace polynomials $M_m(X)$ and cycle polynomials $N_\lambda(X)$; we recall facts about them below.

2.1. Necklace Polynomials. For each degree $m \geq 1$ we first define the m -th necklace polynomial $M_m(X)$ by

$$M_m(X) := \frac{1}{m} \sum_{d|m} \mu(d) X^{m/d}. \quad (2.1)$$

where $\mu(d)$ is the Möbius function. One has $M_1(X) = X$, $M_2(X) = \frac{1}{2}(X^2 - X)$, $M_3(X) = \frac{1}{3}(X^3 - X)$ and $M_6(X) = \frac{1}{6}(X^6 - X^3 - X^2 + X)$. The polynomial $M_m(X)$ has rational coefficients but takes integer values at integers. For a positive integer n the (positive integer) value $M_m(n)$ has a combinatorial interpretation as counting the number of different necklaces having n distinct colored beads taking at most n colors, which have the property of being *primitive* in the sense that their cyclic rotations are distinct, as noted in 1872 by Moreau [19]. They were named necklace polynomials in Metropolis and Rota [18].

The necklace polynomials at values $z = q = p^f$ count monic degree n irreducible polynomials over finite fields \mathbb{F}_q .

Proposition 2.1. *Fix a prime $p \geq 2$, and let $q = p^f$. For each $n \geq 1$ consider the set $\mathcal{F}_{n,q}$ of all monic degree n polynomials*

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{F}_q[X].$$

Let $N_n^{\text{irred}}(\mathbb{F}_q)$ count the number of irreducible polynomials in $\mathcal{F}_{n,q}$. Then

$$N_n^{\text{irred}}(\mathbb{F}_q) = M_n(q),$$

where $M_n(X)$ is the n -th necklace polynomial.

Proof. This very well-known formula goes back to Gauss, as discussed in Frei [9]. A proof appears in Rosen [22, p. 13]. \square

The following result gives bounds on the size of necklace polynomial values ([13, Lemma 4.2]).

Lemma 2.2. (1) *The necklace polynomial $M_m(X)$ has $M_m(0) = 0$ for $m \geq 1$ and*

$$M_m(1) = \begin{cases} 1 & \text{for } m = 1, \\ 0 & \text{for } m \geq 2. \end{cases}$$

In addition $(X - 1)^2 \nmid M_m(X)$ for all $m \geq 2$.

(2) *One has*

$$M_m(t) > 0, \text{ for all real } t \geq 2.$$

In addition, for $1 \leq j \leq m$ there holds for real $t > m - 1$,

$$M_j(t) > \left\lfloor \frac{m}{j} \right\rfloor - 1. \quad (2.2)$$

(3) *For $m \geq 1$ one has*

$$(-1)^m M_m(-t) > 0, \text{ for all real } t \geq 2. \quad (2.3)$$

In addition, for each $m \geq 2$ and $t > 0$ with $t(t + 1) > m - 2$, there holds for $1 \leq j \leq m/2$,

$$M_{2j}(t) > \left\lfloor \frac{m}{2j} \right\rfloor - 1. \quad (2.4)$$

2.2. Cycle Polynomials. For each partition λ of n we define the *cycle polynomial* $N_\lambda(X) \in \mathbb{Q}[X]$, given by

$$N_\lambda(X) := \prod_{j=1}^n \binom{M_j(X)}{c_j(\lambda)} \quad (2.5)$$

It is a polynomial of degree n since $\sum_{j=1}^n j c_j = n$.

The following result relates factorization of monic polynomials over \mathbb{F}_q for $q = p^k$ with given cycle structure to values of the cycle polynomials at $X = q$.

Proposition 2.3. *Fix a prime $p \geq 2$, and let $q = p^f$. Let $\mathcal{F}_{n,q}$ denote the set of all monic degree n polynomials with coefficients in \mathbb{F}_q , which has cardinality $|\mathcal{F}_{n,q}| = q^n$. Then:*

(1) *Exactly $q^n - q^{n-1}$ polynomials in $\mathcal{F}_{n,q}$ are square-free when factored into irreducible factors over $\mathbb{F}_q[X]$. Equivalently, the probability of a uniformly drawn random polynomial in $\mathcal{F}_{n,q}$ hitting the discriminant locus $\text{Disc}(f(X)) = 0$ is exactly $\frac{1}{q}$.*

(2) *The number $N_\lambda^*(q)$ of $f(x) \in \mathcal{F}_{n,q}$ whose factorization over \mathbb{F}_q into irreducible factors is square-free with factors having degree type $\lambda := (\lambda_1, \dots, \lambda_r)$, with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ with $\sum \lambda_i = n$, having $m_j = m_j(\lambda)$ factors of degree j satisfies*

$$N_\lambda^*(q) = \prod_{j=1}^n \binom{M_j(q)}{m_j(\lambda)}. \quad (2.6)$$

Thus $N_\lambda^*(q) = N_\lambda(q)$, the cycle polynomial $N_\lambda(X)$ evaluated at $X = q$.

Proof. (1) This result follows from [22, Prop. 2.3]. Another proof, due to M. Zieve, is given in [30, Lemma 4.1].

(2) This equality of $N_\lambda^*(q)$ to this product is well known, see for example S. R. Cohen [3, p. 256]. It follows from counting all unique factorizations of the given type. \square

Cycle polynomials have the following properties.

Lemma 2.4. (Properties of Cycle Polynomials) *Let $n \geq 2$. For any partition λ of n the cycle polynomial $N_\lambda(X)$ has the following properties:*

- (1) *The polynomial $N_\lambda(X) \in \frac{1}{n!}\mathbb{Z}[X]$ is integer-valued.*
- (2) *The polynomial $N_\lambda(X)$ has lead term*

$$\left(\prod_{j=1}^n \frac{1}{j^{c_j(\lambda)} c_j(\lambda)!} \right) X^n = \frac{|C_\lambda|}{n!} X^n.$$

- (3) *The polynomial $N_\lambda(X)$ is divisible by X^m , where $m \geq 1$ counts the number of distinct cycle lengths appearing in λ .*

Proof. These properties are proved in [13, Lemma 4.3]. \square

The following result for divisibility of the cycle polynomial $N_\lambda(X)$ by $X - 1$ is the source of the ‘‘Springer regular element’’ property.

Lemma 2.5. (Divisibility of Cycle Polynomials by $X - 1$)

- (1) *Let $n \geq 2$. For any partition λ of n the cycle polynomial $N_\lambda(X)$ is divisible by $X - 1$.*

- (2) *Such a polynomial is divisible by $(X - 1)^2$ if and only if the partition λ has at least two distinct parts $\lambda_i > \lambda_j \geq 2$ or else has a part $\lambda_i > 1$ and at least two parts equal to 1.*

Proof. Lemma 2.2(1) says that $M_m(X)$ for $m \geq 2$ contains a factor of $X - 1$. In this case $(X - 1) \mid \binom{M_m(X)}{k}$ for any $k \geq 1$. The only N_λ not covered by this result are those with $\lambda = [1^n]$.

For the “if” direction of (2) for $n \geq 2$ one has $(X - 1) \mid N_{[1^n]} = \binom{M_1(X)}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$. This gives a sufficient condition for $(X - 1)$ to divide two different factors in the product (2.6) defining $N_\lambda(X)$. For the “only if” part of (2) we see that the remaining partitions either have the form $[b^a]$ with $ab = n$ or else have the form $[d^c, 1]$ with $cd = n - 1$. We must show $(X - 1) \mid N_\lambda(X)$ in these cases. For the case $[b^a]$ we have $M_m(1) = 0$, and we have $M'_m(1) \neq 0$ by Lemma 2.2 (1). Furthermore $M_m(X) - j$ for $j \geq 0$ has nonzero value j at $X = 1$, so contributes no extra root. So the multiplicity of the factor $(X - 1)$ is 1 in this case. In the remaining case the same argument applies, with the extra factor $M_1(X) = X$ being nonzero at $X = 1$. \square

There is a simple formula giving the sum of all polynomials $N_\lambda(X)$ over all partitions λ of n .

Lemma 2.6. (Sum of Cycle Polynomials) *For fixed $n \geq 2$ there holds*

$$\sum_{\lambda \vdash n} N_\lambda(X) = X^{n-1}(X - 1). \quad (2.7)$$

Proof. Both sides are polynomials of degree n , so it suffices to check that their values agree at $n + 2$ points. In fact one checks that their values agree at $X = p^f$ for all prime powers p^f using the two parts of Proposition 2.3. \square

2.3. z -splitting measures.

Definition 2.7. (1) *The z -splitting measure $\nu_{n,z}$ is defined on conjugacy classes C_λ of S_n by*

$$\nu_{n,z}^*(C_\lambda) := \frac{1}{z^{n-1}(z - 1)} \prod_{j=1}^n \binom{M_j(z)}{m_j(\lambda)}. \quad (2.8)$$

in which $m_j = m_j(\lambda)$ counts the number of cycles in $g \in S_n$ of length j , and for a complex number z we interpret $\binom{z}{k} := \frac{(z)_k}{k!} = \frac{z(z-1)\cdots(z-k+1)}{k!}$.

(2) *The measure is extended from conjugacy classes to elements $g \in S_n$ by requiring it to be constant within a conjugacy class.*

A well known formula for the size of a conjugacy class states ([26, p. 28]),

$$|C_\lambda| = n! \prod_{j=1}^n \frac{j^{-m_j(\lambda)}}{m_j(\lambda)!}. \quad (2.9)$$

Using it we obtain

$$\nu_{n,z}^*(g) := \frac{1}{n!} \cdot \frac{1}{z^{n-1}(z - 1)} \prod_{j=1}^n j^{m_j(\lambda)} m_j(\lambda)! \binom{M_j(z)}{m_j(\lambda)}. \quad (2.10)$$

In terms of the cycle polynomials $N_\lambda(z)$, the z -splitting measure is given on conjugacy classes of S_n by

$$\nu_{n,z}^*(C_\lambda) := \frac{1}{z^{n-1}(z-1)} N_\lambda(z). \quad (2.11)$$

Lemma 2.5 shows that $(z-1) \mid N_\lambda(z)$, so this measure takes well-defined values at all $z \in \mathbb{C} \setminus \{0\}$.

2.4. Random polynomial splitting interpretation of z -splitting measures. The z -splitting measures at $z = q = p^f$ arise as the splitting probabilities for factorizations of monic degree n polynomials over \mathbb{F}_q , as shown in [13]. Recall that $\mathcal{F}_{n,q}$ denotes the set of all degree n monic polynomials $f(x) \in \mathbb{F}_q[x]$. We can factor a given $f(x)$ uniquely as $f(x) = \prod_{i=1}^k g_i(x)^{e_i}$, where the e_i are positive integers and the $g_i(x)$ are distinct, monic, irreducible, and non-constant. We let $\lambda \vdash n$ denote the partition of n given by the degrees of the factors $g_i(x)$.

Proposition 2.8. *Consider drawing a random monic polynomial $f(X)$ from $\mathcal{F}_{n,q}$ with the uniform distribution. Then the conditional probability of $f(x)$ having a factorization into irreducible factors of splitting type λ , conditioned on $g(x)$ having a square-free factorization, is exactly $\nu_{n,q}^*(C_\lambda)$. That is,*

$$\nu_{n,q}^*(C_\lambda) = \frac{\text{Prob}[f(x) \text{ has splitting type } \lambda \text{ and } f(x) \text{ square-free}]}{\text{Prob}[f(x) \text{ is square-free}]}.$$

Proof. Proposition 2.1 and Proposition 2.3 (1) together give

$$\frac{\text{Prob}[f(x) \text{ has splitting type } \lambda \text{ and } f(x) \text{ square-free}]}{\text{Prob}[f(x) \text{ is square-free}]} = \frac{1}{q^n - q^{n-1}} \prod_{j=1}^n \binom{M_j(q)}{m_j(\lambda)}.$$

Comparison of the right side with the definition (2.8) of the necklace measure shows equality at $z = q$ with $\nu_{n,q}^*(C_\lambda)$. \square

3. SPLITTING MEASURES FOR $z = 1$

The object of this paper is to treat the z -splitting measures when $z = 1$. The well-definedness of the splitting measure $\nu_{n,1}^*(C_\lambda)$ at $z = 1$ follows from the formula (2.11) using the fact that $(X-1) \mid N_\lambda(X)$ for $n \geq 2$. These measures turn out to be (strictly) signed measures for all $n \geq 3$.

3.1. Decomposition into a sum of two measures attached to n and $n-1$. We show that the signed measure $\nu_{n,1}^*$ is supported on a small set of conjugacy classes C_λ and that it can be expressed as a sum of two signed measures ω_n and ω_{n-1}^* , both constructed in terms of a family of auxiliary measures $\{\omega_n : n \geq 1\}$, one for each S_n . That is, ω_{n-1}^* is a measure on S_n obtained from ω_{n-1} on S_{n-1} in a simple fashion described in the following result.

Theorem 3.1. *The signed measures $\nu_{n,1}^*$ have the following properties.*

(1) *The support of the measure $\nu_{n,1}^*$ is exactly the set of conjugacy classes $[\lambda]$ such that λ is one of:*

- (i) Rectangular partitions $\lambda = [b^a]$ for $ab = n$.
- (ii) Almost-rectangular partitions $\lambda = [d^c, 1]$ for $cd = n - 1$.
- (2) The measure $\nu_{n,1}^*$ is a sum of two signed measures on S_n ,

$$\nu_{n,1}^* = \omega_n + \omega_{n-1}^*,$$

which are uniquely characterized for all $n \geq 1$ by the following two properties:

- (i) ω_n is supported on the rectangular partitions $[b^a]$ of S_n ,
- (ii) ω_{n-1}^* is supported on the almost-rectangular partitions of S_n , those of the form $[d^c, 1]$, and is obtained from ω_{n-1} on S_{n-1} , as follows. For $\lambda \vdash n$,

$$\omega_{n-1}^*(C_\lambda) := \begin{cases} \omega_{n-1}(C_{\lambda'}) & \text{if } \lambda = [\lambda', 1] \text{ with } \lambda' \vdash n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The supports of ω_n and ω_{n-1}^* overlap on the identity conjugacy class $\lambda = [1^n]$, viewing $[1^n]$ as being both rectangular and almost-rectangular.

- (3) For $n \geq 2$,

$$\nu_{n,1}^*(C_{[1^n]}) = \frac{(-1)^n}{n(n-1)}.$$

Proof. (1) The support of the 1-splitting measure $\nu_{n,1}^*(C_\lambda)$ consist of all conjugacy classes C_λ for which $(X-1)^2 \nmid N_\lambda(X)$. Lemma 2.4(4) says that this condition holds if and only if either $\lambda = [b^a]$ with $ab = n$ or $\lambda = [d^c, 1]$ with $cd = n - 1$.

- (2) We recursively define $\omega_n(\cdot)$ in terms of $\omega_{n-1}(\cdot)$ by

$$\omega_n(\lambda) := \begin{cases} \nu_{n,1}^*(C_\lambda) & \text{if } \lambda = [b^a], \text{ with } n = ab, b > 1, \\ \nu_{n,1}^*(C_{[1^n]}) - \omega_{n-1}(C_{[1^{n-1}]}) & \text{if } \lambda = [1^n], \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

The initial condition for $n = 1$ is $\omega_1(C_{[1]}) = \nu_{1,1}^*(C_{[1]}) = 1$. Here ω_n satisfies property (i) with this definition, and conversely, property (i) forces uniqueness of the definition on $[b^a]$ with $b > 1$, and uniqueness for the “otherwise” term. Next, property (ii) forces the definition on $[1^n]$, which establishes that the measure ω_n is unique if it exists. The uniqueness of ω_n then forces the uniqueness of ω_{n-1}^* .

It remains to show that this definition has property (ii). We define

$$\omega_{n-1}^*(C_\lambda) := \nu_{n,1}^*(C_\lambda) - \omega_n(C_\lambda). \quad (3.2)$$

By the support condition (1) for $\nu_{n,1}^*$, if $\omega_{n-1}^*(C_\lambda) \neq 0$ then necessarily $\lambda = [d^c, 1]$ where $n - 1 = cd$, with $d > 1$ or with $\lambda = [1^n]$. The recursive definition above also forces

$$\omega_{n-1}^*(C_{[1^n]}) = \omega_{n-1}(C_{[1^{n-1}]}).$$

It remains to prove that property (ii) holds; for all partitions of the form $\lambda = [d^c, 1] \vdash n$ having $d > 1$, there holds

$$\omega_n^*(C_{[d^c, 1]}) = \omega_{n-1}(C_{[d^c]}).$$

By the recursive definition, this identity is equivalent to the assertion that

$$\nu_{n,1}^*(C_{[d^c, 1]}) = \nu_{n-1,1}^*(C_{[d^c]}).$$

This in turn, using (2.11) is equivalent to the assertion, for $n - 1 = cd$ with $d > 1$,

$$\frac{1}{t-1}N_{[d^c,1]}(t)|_{t=1} = \frac{1}{t-1}N_{[d^c]}(t)|_{t=1}. \quad (3.3)$$

Here we have

$$N_{[d^c,1]}(t) = \binom{M_1(t)}{1} N_{[d^c]}(t)$$

and the equality (3.3) follows since $\binom{M_1(t)}{1}|_{t=1} = 1$.

(3) For $n = 1$, $\nu_{1,1}^*([1]) = 1$. For $n \geq 2$, we have

$$\begin{aligned} \nu_{n,1}^*([1^n]) &= \frac{1}{X^n(X-1)} \prod_{i=1}^n \frac{(X-i+1)}{i} \Big|_{X=1} \\ &= \frac{(-1)^{n-2}(n-2)!}{n!} = \frac{(-1)^n}{n(n-1)}. \end{aligned}$$

□

3.2. Structure of the measures ω_n . Theorem 3.1 effectively reduces the study of the 1-splitting measures $\nu_{n,1}^*$ to the study of the family of measures ω_n , which are signed measures for certain n .

Theorem 3.2. *The measure ω_n is given as follows. For $n \geq 2$ and each partition $\lambda \vdash n$,*

$$\omega_n(C_\lambda) = \begin{cases} (-1)^{a+1} \frac{\phi(b)}{n} & \text{if } \lambda = [b^a] \text{ for the factorization } n = ab, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

Proof. By definition the measure ω_n is constant on conjugacy classes and is supported on elements having cycle structure $\lambda = [b^a]$ where $n = ab$. If $b > 1$ then we have $\omega_n(C_{[b^a]}) = \nu_{n,1}^*(C_{[b^a]})$. In this case by Lemma 2.2 (1) gives $(X-1)|M_b(X)$ and also

$$\frac{M_b(X)}{X-1} \Big|_{X=1} = M'_b(1) = \prod_{p|b} \left(1 - \frac{1}{p}\right) = \frac{\phi(b)}{b} > 0,$$

where $\phi(b)$ is Euler's totient function. In addition

$$(M_b(X) - j) \Big|_{X=1} = -j.$$

We obtain

$$\nu_{n,1}^*(C_{[b^a]}) = \frac{1}{a!} \cdot \frac{\phi(b)}{b} \prod_{j=1}^{a-1} (-j) = (-1)^{a-1} \frac{\varphi(b)}{ab} = (-1)^{a+1} \frac{\phi(b)}{n},$$

where $\phi(b)$ is Euler's totient function. Thus for $b > 1$ we obtain

$$\omega_n(C_{[b^a]}) = \nu_{n,1}^*(C_{[b^a]}) = (-1)^{a+1} \frac{\phi(b)}{n}.$$

For the remaining case $b = 1$, where $a = n$, we define for $n = 1$,

$$\omega_1(C_{[1]}) = \nu_{1,1}^*(C_{[1]}) = 1.$$

For $n \geq 2$ we have the recursion (as in Theorem 3.1)

$$\omega_n([1^n]) = \nu_{n,1}^*(C_{[1^n]}) - \omega_n^*(C_{[1^{n-1},1]}) = \nu_{n,1}^*(C_{[1^n]}) - \omega_{n-1}(C_{[1^{n-1}]}).$$

Using the formula of Theorem 3.1 (3) we have

$$\nu_{n,1}^*(C_{[1^n]}) = \frac{(-1)^n}{n(n-1)}.$$

It follows that

$$\omega_2(C_{[1^2]}) = \nu_{2,1}^*(C_{[1^2]}) - \omega_1(C_{[1]}) = \frac{1}{2} - 1 = -\frac{1}{2}.$$

We prove by induction on $n \geq 2$ that

$$\omega_n(C_{[1^n]}) = \frac{(-1)^{n+1}}{n}. \quad (3.5)$$

This result is equivalent to the identity

$$\frac{(-1)^{n+1}}{n} = \frac{(-1)^n}{n(n-1)} - \frac{(-1)^n}{n-1},$$

which is

$$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}.$$

The formula (3.5) matches the theorem's formula (3.4) for $b = 1$. \square

We use the explicit description of the measures ω_n to derive the following consequences about their structure.

Theorem 3.3. (Structure of measures ω_n) *For $n \geq 2$ the measures ω_n on S_n have the following properties.*

(1) *Each measure ω_{2m+1} is nonnegative and has total mass 1, so is a probability measure. It is supported on even permutations, and its restriction $\omega_n|_{A_n}$ to the alternating group A_n is a probability measure.*

(2) *Each measure ω_{2m} is a signed measure having total signed mass 0. The measure of ω_n is nonnegative on odd permutations, having total mass $\frac{1}{2}$. It is non-positive on even permutations, having total signed mass $-\frac{1}{2}$. The absolute value measure $|\omega_{2m}|$ is a probability measure, and the measure $-2\omega_{2m}|_{A_{2m}}$ restricted to the alternating group A_{2m} is a probability measure.*

(3) *The family of all measures ω_n have a product structure compatible with multiplication of integers. Namely, setting $n = \prod_i p_i^{e_i}$, and for any factorization $ab = n$, there holds*

$$\omega_n(C_{[b^a]}) = \prod_i \omega_{p_i^{e_i}}(C_{[(b_i)^{a_i}]}), \quad (3.6)$$

in which $b_i = p_i^{e_{i,2}}$ (resp. $a_i = p_i^{e_{i,1}}$) represent the maximal power of p_i dividing b (resp. a), so that $e_{i,1} + e_{i,2} = e_i$. We allow some or all values $e_{i,j} = 0$ for $j = 1, 2$, so $b = 1$ or $a = 1$ is allowed.

Proof. (1) Suppose first that $n = 2m + 1$ is odd. Then $n = ab$ has both a, b odd. Therefore all $\omega_n(C_{[b^a]}) > 0$ in (3.4), and we conclude that ω_{2m+1} is a nonnegative measure. In addition all permutations of cycle shape $[b^a]$ are even permutations, so the support of ω_{2m+1} is inside the alternating group A_{2m+1} .

We set for all $n \geq 1$ the signed mass

$$\mathfrak{m}_n := \sum_{g \in S_n} \omega_n(g) = \sum_{\lambda \vdash n} \omega_n(C_\lambda).$$

To show that ω_{2m+1} is a probability measure, we wish to show $\mathfrak{m}_{2m+1} = 1$. We claim that $\mathfrak{m}_{2m} = 0$ and $\mathfrak{m}_{2m+1} = 1$ for all $m \geq 1$. We have $\mathfrak{m}_1 = 1$. We have for all $n \geq 2$ the relation

$$\begin{aligned} 1 &= \sum_{g \in S_n} \nu_{n,1}^*(g) \\ &= \sum_{\lambda \vdash n} \omega_n(C_\lambda) + \sum_{\lambda \vdash n} \omega_n^*(C_\lambda) \\ &= \mathfrak{m}_n + \sum_{\lambda' \vdash n-1} \omega_{n-1}(C_{\lambda'}) = \mathfrak{m}_n + \mathfrak{m}_{n-1}, \end{aligned}$$

since only elements of form $\lambda = [\lambda', 1]$ contribute in the second sum. The relation $\mathfrak{m}_n + \mathfrak{m}_{n-1} = 1$ now yields by induction on $m \geq 1$ that each $\mathfrak{m}_{2m} = 0$ and each $\mathfrak{m}_{2m+1} = 1$. In particular, ω_{2m+1} is a probability measure. This proves assertion (1).

(2) Suppose $n = 2m$ is even. A permutation g of cycle type $[b^a]$ is an odd permutation if and only if the integer a is odd, and b is even. This condition holds exactly when a is odd, and in this case $(-1)^{a+1} = 1$. In consequence such a permutation is an even permutation if and only if a is even, in which case $(-1)^{a+1} = -1$. It follows by Theorem 3.2 that the measure is nonnegative on odd permutations and is nonpositive on even permutations.

The argument in (1) showed that ω_{2m} has total mass $\mathfrak{m}_{2m} = 0$. Now we have, for any $n \geq 1$,

$$\sum_{\lambda} |\omega_n|(C_\lambda) = \sum_{ab=n} |\omega_n|(C_{[b^a]}) = \sum_{ab=n} \frac{\phi(b)}{n} = \frac{1}{n} \left(\sum_{b|n} \phi(b) \right) = 1.$$

It follows that $|\omega_n|$ is a probability measure for all $n \geq 1$. From $\mathfrak{m}_{2m} = 0$ it follows that for even n the positive elements have total mass $1/2$ and the negative elements have total mass $1/2$.

We have seen that all the negative weight elements lie in the alternating group A_{2m} , and the positive weight elements all lie in $S_{2m} \setminus A_{2m}$. The conclusions on $|\omega_{2m}|$ and $-2\omega_{2m}|_{A_{2m}}$ follow. This proves assertion (2).

(3) Note first that in the factorization formula (3.6) with $b = \prod_i p_i^{e_{i,2}} \geq 1$ all factors at odd primes p_i on the right are nonnegative. Thus only the factor at $p_i = 2$ can contribute to the sign of $\omega_n(C_{[b^a]})$.

The factorization formula is easily verified by direct calculation using the formula for ω_n of Theorem 3.2. Ignoring signs, it asserts

$$\frac{\phi(b)}{n} = \prod_i \frac{\phi(p_i^{e_i,2})}{p_i^{e_i}},$$

which holds by multiplicativity of the Euler totient function. To verify the sign condition, note that it automatically holds if n is odd. If n is even, and $p_1 = 2$, then $\omega_n(C_{[b^a]}) > 0$ if and only if a is odd. This holds if and only if $e_{1,1} = 0$, which holds if and only if a_1 is odd, which holds if and only if $\omega_{2^{e_1}}(C_{[b_1^{a_1}]}) > 0$. \square

The explicit formula for the measure ω_n has the following consequences for the splitting measure $\nu_{n,1}^*$.

Theorem 3.4. (Properties of 1-splitting measures) *The 1-splitting measures $\nu_{n,1}^*$ have the following properties.*

(i) *The splitting measure $\nu_{n,1}^*$ is nonnegative on odd permutations $S_n \setminus A_n$ and there has total mass $\frac{1}{2}$. It is a signed measure on even permutations A_n and there has total (signed) mass $\frac{1}{2}$.*

(ii) *The absolute value splitting measure $|\nu_{n,1}^*|$ has total mass 2, with total mass $\frac{1}{2}$ on odd permutations and total mass $\frac{3}{2}$ on even permutations.*

Proof. Assertion (i) follows from Theorem 3.3 (i) and (ii). Here the sign of $[d^c, 1]$ in S_n agrees with the sign of $[d^c]$ in S_{n-1} . On odd permutations ω_n and ω_{n-1} are nonnegative, and restricted to them one measure has total mass 0 and the other total mass $\frac{1}{2}$. On even permutations one of them is nonnegative with total mass 1 and the other nonpositive with total mass $-\frac{1}{2}$.

Assertion (ii) follows from the proof of assertion (i). \square

4. PROBABILISTIC CHARACTERIZATION OF POSITIVE MEASURES $|\omega_n|$ AND SIGNED MEASURES ω_n AT $z = 1$

In this section we show there is an alternate description of the absolute probability measure $|\omega_n|$ given as the output of a probabilistic sampling method. Additionally we give a sampling description to draw random signed elements of the signed measure ω_n .

Random power of n -cycle distribution.

- (1) Draw an n -cycle g from S_n uniformly, with probability $\frac{1}{(n-1)!}$ for each n -cycle.
- (2) Draw an integer $1 \leq j \leq n$ uniformly with probability $\frac{1}{n}$, independently of the draw of g .
- (3) Set $h = g^j$. Take the induced distribution of h on the elements of S_n .

Theorem 4.1. *The absolute value measure $|\omega_n|$ on S_n is a probability measure that coincides with that given by the random power n -cycle distribution.*

Proof. We let ω_n^D denote the random power of n -cycle distribution. Both distributions are constant on conjugacy classes. It suffices to check that the probabilities of this distribution agree with those inferred from Theorem 3.2, which for $n = ab$ are

$$|\omega_n|(C_{[b^a]}) = \frac{\phi(b)}{n},$$

and which are 0 on all other conjugacy classes C_λ .

The cycle structure of g is $[n]$ and that of $h = g^j$ is of the form $[b^a]$ where $n = ab$ with $a = \gcd(j, n)$. Therefore the distribution ω^D is supported on the conjugacy classes of form $C_{[b^a]}$, and we note that all elements in a conjugacy class are assigned the same probability.

The probability density is determined entirely by the value of $a = \gcd(j, n)$, which specifies both a and b . This divisibility condition factorizes over prime powers $n = \prod_i p_i^{e_i}$. For $1 \leq k \leq e_i$, the probability that p_i^k divides a randomly drawn $j \in [1, n]$, so that p_i^k divides $a = \gcd(j, n)$, is $\frac{1}{p_i^k}$. Therefore the probability that p_i^k exactly divides $\gcd(j, n)$ is $\frac{1}{p_i^k}(1 - \frac{1}{p_i})$ if $k < e_i$ and is $\frac{1}{p_i^{e_i}}$, if $k = e_i$. On the other hand, $p_i^{e_i-k}$ exactly divides b so that this probability equals $\frac{\phi(p_i^{e_i-k})}{p_i^{e_i}}$. We deduce that

$$|\omega_n^D|(C_{[b^a]}) = \frac{\phi(b)}{n},$$

giving the desired equality. \square

There is an analogous probabilistic sampling description of the signed measures ω_n for $n \geq 2$.

Signed random power of n -cycle distribution.

- (1) Draw an n -cycle g from S_n uniformly, with probability $\frac{1}{(n-1)!}$ for each n -cycle.
- (2) Draw an integer $1 \leq j \leq n$ uniformly, independently of the draw of g , probability $\frac{1}{n}$.
- (3) Set $h = g^j$. Assign to h its sign $\text{sgn}(h) = (\text{sgn}(g))^j = (-1)^{(n+1)j}$. Take the induced signed distribution of h on the elements of S_n .

This distribution gives something new only when $n = 2m$ is even, and is the unsigned distribution above if $n = 2m + 1$ is odd.

Theorem 4.2. *Let n be even. The measure ω_n on S_n coincides with that given by the signed random power n -cycle distribution.*

Proof. We suppose $n = 2m$ is even, and we let ω_n^{SD} denote the signed random power of n -cycle distribution. We check that the probabilities ω_n^{SD} agree with those given in Theorem 3.2. The cycle structure of $h = g^j$ is of the form $[b^a]$ where $n = ab$ with $a = \gcd(j, n)$. If a is even then $\text{sgn}(h) = (\text{sgn}(g))^a = 1$, while if a is odd then b is even and $\text{sgn}(h) = (\text{sgn}(g))^a = \text{sgn}(g) = -1$. It follows that

$$\omega_n^{SD}(C_{[b^a]}) = (-1)^{a+1} \omega_n^D(C_{[b^a]}).$$

By Theorem 3.2 we have

$$\omega_n^D([b^a]) = |\omega_n|(C_{[b^a]}) = \frac{\phi(b)}{n}$$

whence

$$\omega_n^{SD}(C_{[b^a]}) = (-1)^{a+1} \frac{\phi(b)}{n} = \omega_n(C_{[b^a]}).$$

as asserted. \square

5. REPRESENTATION-THEORETIC INTERPRETATION OF ω_n

The measures ω_n (resp. $|\omega_n|$) are class functions on S_n , so they can be viewed as rational linear combinations of irreducible characters of S_n .

We show that if one rescales these measures by the factor $n!$, which is the smallest positive factor arranging that all character values become integers, then $n!\omega_n$ is the character of a *genuine* virtual representation, one which is an integral linear combination of irreducible representations. We show also that $n!|\omega_n|$ is the character of a genuine representation.

5.1. The measure $n!|\omega_n|$ is the character of a genuine representation of S_n .

Theorem 5.1. *For all $n \geq 1$ the class function $n!|\omega_n|(\cdot)$ is the character of the induced representation*

$$\rho_n^+ := \text{Ind}_{C_n}^{S_n}(\chi_{\text{triv}}),$$

from the cyclic group C_n generated by an n -cycle of S_n , carrying the trivial representation χ_{triv} . The representation ρ is of degree $(n-1)!$ and for $n \geq 3$ is a reducible representation. The trivial representation occurs in ρ_n^+ with multiplicity one and the sign representation χ_{sgn} occurs with multiplicity 1 if n is odd and multiplicity 0 if n is even.

Proof. We let $C_n = \langle h \rangle$ denote the cyclic subgroup of S_n represented by the n -cycle $h = (123 \cdots n)$, and let χ_{triv} denote the trivial one-dimensional representation. The cycle structure of h^k for $1 \leq k \leq n$ is $[b^a]$ where $b = n/\gcd(n, k)$. In particular there are $\phi(b)$ elements of C_n having cycle structure $[b^a]$, for each $b|n$.

The induced representation $\rho_n^+ = \text{Ind}_{C_n}^{S_n}(\chi_{\text{triv}})$ is a permutation representation of degree $(n-1)!$. We compute its character $\psi := \psi_\rho$ (with $\rho = \rho^+$) using the Frobenius formula for the character of an induced representation $\psi(g) = \text{Tr}(\text{Ind}_H^G(\sigma)(g))$, in terms of the character $\chi(h) = \text{Tr}(\sigma(h))$ of the original representation σ on H , cf. [11, Sect. 3.3]. Applied to the case here it states

$$\psi(g) = \sum_{x \in S_n/C_n} \hat{\chi}(x^{-1}gx),$$

in which

$$\hat{\chi}(g) = \begin{cases} \chi(x^{-1}gx) & \text{if } x^{-1}gx \in H \\ 0 & \text{otherwise} \end{cases} \quad (5.1)$$

We can also eliminate the cosets and write for any subgroup

$$\psi(g) = \frac{1}{|H|} \sum_{x \in S_n} \widehat{\chi}(x^{-1}gx), \quad (5.2)$$

In the case $H = C_n$ and χ_{triv} we have

$$\psi(g) = \frac{1}{n} \sum_{x \in S_n} \widehat{\chi}_{triv}(x^{-1}gx), \quad (5.3)$$

For a conjugacy class C_λ we set

$$\psi(C_\lambda) = \sum_{g \in C_\lambda} \psi(g) = |C_\lambda| \psi(g).$$

Clearly $\psi(g) = 0$ if g is not conjugate to some element of C_n , so we have

$$\psi(C_\lambda) = 0 \quad \text{when} \quad \lambda \neq [b^a] \quad \text{for any } ab = n.$$

For the exceptional case, the formula (5.3) counts

$$\psi(C_{[b^a]}) = |\{(g', x, h) : h = xg'x^{-1} \text{ with } g' \in C_{[b^a]}, h \in C_n, x \in S_n\}|.$$

There are $|C_{[b^a]}| = \frac{n!}{b^a a!}$ choices for g , there are $\phi(b)$ choices for h , and for each such pair there are $|N(\langle g' \rangle)| = b^a a!$ choices of x , in which $|N(G)|$ denotes the cardinality of the normalizer of the subgroup G in S_n . Here $G = C_n$ is a cyclic group of order b generated by an element g' having cycle structure $[b^a]$. We obtain

$$\psi(C_{[b^a]}) = \frac{1}{n} \left(\frac{n!}{b^a a!} \cdot b^a a! \cdot \phi(b) \right) = n! \frac{\phi(b)}{n}.$$

On comparing this character with the formula for the class function $n!|\omega_n|(C_{[b^a]})$ implied by Theorem 3.2 we find agreement

$$n!|\omega_n|(C_\lambda) = \psi_\rho(C_\lambda), \quad \text{for all } \lambda \vdash n.$$

Now that we know the class function $n!|\omega_n|$ is the character ψ_n^+ of a genuine representation ρ_n^+ , we may write it in terms of the basis of irreducible characters as

$$\psi_n^+ = n!|\omega_n| = \sum_{\pi \in \text{Irr}(S_n)} m(\pi; n!|\omega_n|) \pi.$$

with nonnegative integer multiplicities $m(\pi; n!|\omega_n|)$. For any nonzero class function $f(\cdot)$ the (signed) multiplicity is the real number

$$m(\pi; f) := \frac{1}{\pi(1)} \langle f, \pi \rangle := \frac{1}{\pi(1)n!} \sum_{g \in S_n} f(g) \pi(g)$$

We know by Theorem 3.3 that the total mass $|\omega_n|(S_n)$ is 1 so that

$$\chi_{\rho_n^+}(S_n) = n!|\omega_n|(S_n) = n! \left(\sum_{g \in S_n} |\omega_n|(g) \right) = n!.$$

Now $\chi_{triv}(S_n) = n!$ whence the multiplicity of the trivial representation in $|\omega_n|$ is

$$m(\chi_{triv}; \rho_n^+) = m(\chi_{triv}; n!|\omega_n|) = \frac{\langle \chi_{triv}, n!|\omega_n| \rangle}{\langle \chi_{triv}, \chi_{triv} \rangle} = 1.$$

Since the representation ρ_n^+ has degree $(n-1)!$ and a summand of degree 1 it must be reducible for all $n \geq 3$.

Now consider the sign representation χ_{sgn} . By Theorem 3.3(ii) all the positive values of the function $(2m)!\omega_{2m}$ are on odd permutations and all the negative values are taken on even permutations, and the function $(2m)!|\omega_{2m}|$ has the same mass taken over all odd permutations versus over all even permutations. Thus

$$m(\chi_{\text{sgn}}; \rho_{2m}^+) = m(\chi_{\text{sgn}}; (2m)!|\omega_{2m}|) = \frac{\langle \chi_{\text{triv}}, -(2m)!\omega_{2m} \rangle}{\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle} = 0,$$

since $\langle \chi_{\text{triv}}, \omega_{2m} \rangle = \sum_{g \in S_{2m}} \omega_{2m}(g) = 0$. \square

5.2. The measure $-(2m)!\omega_{2m}$ is the character of a genuine representation of S_{2m} . Recall that if $n = 2m + 1$ is odd then $\omega_{2m+1} = |\omega_{2m+1}|$, which is handled by Theorem 5.1.

Theorem 5.2. *If $n = 2m$ is even then the class function $-n!\omega_n$ is the character of the induced representation*

$$\rho_{2m}^- := \text{Ind}_{C_{2m}}^{S_{2m}}(\chi_{\text{sgn}})$$

from the cyclic group C_{2m} of a $2m$ -cycle, carrying on it the sign representation. The representation ρ_{2m}^- is of degree $(2m-1)!$ and is a reducible representation for $m \geq 1$. The trivial representation χ_{triv} of S_{2m} occurs in ρ_{2m}^- with multiplicity 0 and the sign representation χ_{sgn} occurs with multiplicity 1.

Proof. Let C_{2m} be a cyclic subgroup of S_{2m} generated by a $(2m)$ -cycle. The induced representation $\rho_n^- = \text{Ind}_{C_{2m}}^{S_{2m}}(\chi_{\text{sgn}})$ is a representation of degree $(n-1)!$, since the sign character on C_{2m} is obtained by restriction from χ_{sgn} and is a representation of degree 1.

We compute the character ψ_n^- of ρ_n^- again using the Frobenius formula for the character of an induced representation,

$$\psi_n^-(g) = \sum_{x \in S_n/C_n} \hat{\chi}_{\text{sgn}}(x^{-1}gx).$$

The sign character is constant on every nonzero term in this sum, so that we have

$$\psi_n^-(g) = \chi_{\text{sgn}}(g) \left(\sum_{x \in S_{2m}/C_{2m}} \hat{\chi}_{\text{triv}}(x^{-1}gx) \right) = \chi_{\text{sgn}}(g) \psi_n^+(g).$$

Since $\psi_n^-(g)$ is a class function we conclude using Theorem 5.1 that

$$\psi_n^-(C_\lambda) = \chi_{\text{sgn}}(g) \psi_n^+(C_\lambda) = \chi_{\text{sgn}}(g) (2m)!|\omega_{2m}|(C_\lambda).$$

Now for $g \in C_{[b^a]}$ we have $\chi_{\text{sgn}}(g) = (-1)^a$ and so we obtain from Theorem 3.2 for $|\omega_{2m}|(C_{[b^a]}) = \frac{\phi(b)}{2m}$ that

$$\psi_n^-(C_\lambda) = \begin{cases} (2m)!(-1)^a \frac{\phi(b)}{2m} & \text{if } \lambda = [b^a] \text{ with } ab = 2m, \\ 0 & \text{otherwise.} \end{cases}$$

Comparison with Theorem 3.2 yields

$$\psi_n^-(C_\lambda) = -(2m)!\omega_{2m}(C_\lambda),$$

as asserted.

Since the total mass of $\omega_{2m}(S_n) = 0$, the multiplicity of the trivial representation in ρ_{2m}^- is

$$m(\chi_{\text{triv}}; \rho_{2m}^-) = \frac{1}{(2m)!} \sum_{g \in S_{2m}} \chi_{\text{triv}}(g) \psi_\rho(g) = \sum_{g \in S_{2m}} -\omega_{2m}(g) = 0.$$

Next for $n = 2m \geq 2$ we determine the multiplicity of the sign representation to be

$$\begin{aligned} m(\chi_{\text{sgn}}; \rho_{2m}^-) &= \frac{1}{(2m)!} \sum_{g \in S_{2m}} \chi_{\text{sgn}}(g) \psi_n^-(g) \\ &= \sum_{\substack{g \in S_{2m} \\ g \in C_{[b^a]}}} (-1)^a (-\omega_{2m}(g)) \\ &= \sum_{g \in S_{2m}} |\omega|_{2m}(g) = 1. \end{aligned}$$

Since ρ_{2m}^- on S_{2m} has the sign representation as a constituent, it is reducible for all $m \geq 1$. \square

5.3. Multiplicities of Representations. There are some simple symmetries of multiplicities of irreducible representations π in ρ_n^\pm with respect to the sign character.

Theorem 5.3. *Let π denote an irreducible representation of S_n , and ψ_π its character, and define*

$$\rho_n^+ := \text{Ind}_{C_n}^{S_n}(\chi_{\text{triv}}) \quad \text{and} \quad \rho_n^- := \text{Ind}_{C_n}^{S_n}(\chi_{\text{sgn}}).$$

Then:

(1) *If $n = 2m + 1$ is odd then $\rho_{2m+1}^+ = \rho_{2m+1}^-$, and the multiplicity*

$$m(\pi; \rho_{2m+1}^+) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m+1}^+). \quad (5.4)$$

(2) *If $n = 2m$ is even then the multiplicity*

$$m(\pi; \rho_{2m}^+) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m}^-). \quad (5.5)$$

Proof. (1) Suppose $n = 2m + 1$ is odd. Then by Theorem 3.3 (i) the character $\psi_\rho = (2m + 1)!\omega_{2m+1}$ is supported on even permutations, and $\omega_{2m+1} = |\omega_{2m+1}|$.

If π is any irreducible representation on S_{2m+1} then

$$\begin{aligned}
m(\pi; \rho_{2m+1}^+) &= m(\pi; (2m+1)!\omega_{2m+1}) \\
&= \frac{\langle \chi_\pi, (2m+1)!|\omega_{2m+1}| \rangle}{\langle \chi_\pi, \chi_\pi \rangle} \\
&= \frac{\langle \chi_\pi \chi_{\text{sgn}}, (2m+1)!|\omega_{2m+1}| \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \chi_{\text{sgn}} \rangle} \\
&= m(\pi \otimes \chi_{\text{sgn}}; \omega_{2m+1}).
\end{aligned}$$

(2) Suppose now that $n = 2m$ is even. Then by Theorem 3.3(2) the character $\psi_\rho = (2m)!\omega_{2m}$ is positive on odd permutations and negative on even permutations, and has equal mass $\frac{1}{2}(2m)!$ on each set. If π is any irreducible representation on S_{2m} then

$$\begin{aligned}
m(\pi; \rho_{2m}^+) &= \frac{\langle \chi_\pi, (2m)!|\omega_{2m}| \rangle}{\langle \chi_\pi, \chi_\pi \rangle} \\
&= \frac{\langle \chi_\pi \chi_{\text{sgn}}, \chi_{\text{sgn}}(2m)!|\omega_{2m}| \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \chi_{\text{sgn}} \rangle} \\
&= \frac{\langle \chi_\pi \chi_{\text{sgn}}, -(2m)! \omega_{2m} \rangle}{\langle \chi_\pi \chi_{\text{sgn}}, \chi_\pi \chi_{\text{sgn}} \rangle} \\
&= m(\pi \otimes \chi_{\text{sgn}}; -(2m)! \omega_{2m}) = m(\pi \otimes \chi_{\text{sgn}}; \rho_{2m}^-).
\end{aligned}$$

□

5.4. The measure $(-1)^n n! \omega_{n-1}^*$ is the character of a genuine representation of S_n . We show that a suitably rescaled version of the measure ω_{n-1}^* in Theorem 3.1 is the character of a genuine representation of S_n which is supported on conjugacy classes of shape $[d^c, 1]$ with $cd = n - 1$.

Theorem 5.4. (1) For each $n \geq 2$ the class function $(-1)^n n! \omega_{n-1}^*$ is the character ψ_L of the induced representation

$$\rho_n^{(L)} := \text{Ind}_{C_{n-1}}^{S_n} ((\chi_{\text{sgn}})^n),$$

viewing $C_{n-1} \subset S_n$ as an $(n-1)$ -cycle holding the symbol n fixed. The representation $\rho_n^{(L)}$ is of degree $n(n-2)!$ and for $n \geq 2$ is a reducible representation.

(2) This representation $\rho_n^{(L)}$ is also given as the induced representation

$$\rho_n^{(L)} = \text{Ind}_{S_{n-1}}^{S_n} (\rho_{n-1}^\epsilon),$$

in which:

- (i) The representation ρ_{n-1}^ϵ with $\epsilon = (-1)^n$ is the genuine representation of S_{n-1} having character $(-1)^n (n-1)! \omega_{n-1}$.
- (ii) S_{n-1} is embedded in S_n as the set of permutations leaving the last symbol n fixed.

Proof. (1) We treat the cases of n even and n odd separately.

Suppose first that $n = 2m$ is even. In this case (1) asserts

$$\rho_n^{(L)} := \text{Ind}_{C_{n-1}}^{S_n}(\chi_{\text{triv}}),$$

with $C_n = \langle h \rangle$ with $h \in S_n$ being a fixed $(n-1)$ -cycle leaving letter n fixed, say $h = (123 \cdots n-2 \ n-1)(n)$. It suffices to compute the character $\psi_{2m}^{(L)}$ of $\rho_{2m}^{(L)} := \text{Ind}_{C_{2m-1}}^{S_{2m}}(\chi_{\text{triv}})$ using the Frobenius formula for the character, since the character determines a (genuine) representation. We have

$$\psi_{2m}^{(L)}(g) = \frac{1}{|C_{2m-1}|} \sum_{x \in S_{2m}} \widehat{\chi}(x^{-1}gx),$$

where $\widehat{\chi}$ is given by (5.1). The powers h^k have cycle structure $[d^c, 1]$ with $cd = 2m-1$, so the character $\psi_{2m}^{(L)}$ is 0 away from these conjugacy classes. The conjugacy class $C_{[d^c, 1]}$ is of size $\frac{(2m)!}{d^c c!}$ by (2.9). We deduce

$$\psi_{2m}^{(L)}(C_{[d^c, 1]}) := \sum_{g \in C_{[d^c, 1]}} \left(\frac{1}{2m-1} \sum_{x \in S_{2m}} \widehat{\chi}(x^{-1}gx) \right)$$

As in Theorem 5.1 we have

$$\psi_{2m}^{(L)}(C_{[d^c, 1]}) = |\{(g', x, h^j) : h = xg'x^{-1} \text{ with } g' \in C_{[d^c, 1]}, h^j \in C_{n-1}, x \in S_n\}|.$$

There are $|C_{[d^c, 1]}| = \frac{(2m)!}{d^c c!}$ choices for g , there are $\phi(d)$ choices for h^j , and for each such pair there are $|N(\langle h^j \rangle)| = d^c c!$ choices of x , where again $|N(G)|$ denotes the cardinality of the normalizer of the subgroup G of S_{2m} . Here G is a cyclic group of order d generated by an element h^j having cycle structure $[d^c, 1]$. We obtain

$$\psi_{2m}^{(L)}(C_{[d^c, 1]}) = \frac{1}{2m-1} \left(\frac{(2m)!}{d^c c!} \cdot d^c c! \cdot \phi(d) \right) = (2m)! \frac{\phi(d)}{2m-1}.$$

On comparing this character with the formula for the class function $(2m)! \omega_{2m-1}(C_{[d^c]})$ implied by Theorem 3.2 and the fact that $\omega_{2m+1}(C_{\lambda'}) = 0$ for non-rectangular partitions, we find

$$\psi_{2m}^{(L)}(C_{[\lambda', 1]}) = (2m)! \omega_{2m-1}(C_{\lambda'}) \quad \text{for all } \lambda' \vdash 2m-1.$$

In addition $\psi_{2m}^{(L)}(C_{\lambda}) = 0$ for all $\lambda \vdash 2m$ with no parts equal to 1, so we have

$$\psi_{2m}^{(L)}(C_{\lambda}) = (2m)! \omega_{2m-1}^*(C_{\lambda}) \quad \text{for all } \lambda \vdash 2m,$$

as asserted. Here $\epsilon = 1$ and the character $\psi_{2m}^{(L)}$ is nonnegative.

Suppose secondly that $n = 2m+1$ is odd, in which case (1) asserts

$$\rho_n^{(L)} := \text{Ind}_{C_{2m}}^{S_{2m+1}}(\chi_{\text{sgn}})$$

We proceed as above. All elements in $C_{[d^c, 1]}$ will be added with the same sign from the character $\chi_{\text{sgn}}(C_{[d^c]}) = (-1)^c$, using the fact that $n-1 = 2m$ is even. We

find

$$\begin{aligned}\psi_{2m+1}^{(L)}(C_{[d^c, 1]}) &= (-1)^a \frac{1}{2m-1} \left(\frac{(2m)!}{d^c c!} \cdot (2m+1)d^c c! \cdot \varphi(d) \right) \\ &= -(2m)!((-1)^{c+1}) \frac{\varphi(d)}{2m-1}.\end{aligned}$$

Comparing the right side with the formula of Theorem 3.2 yields

$$\psi_{2m+1}^{(L)}(C_{[\lambda', 1]}) = -(2m)! \omega_{2m}(C_{\lambda'}) \quad \text{for all } \lambda' \vdash n-1,$$

In addition $\psi_{2m+1}^{(L)}(C_\lambda) = 0$ for all $\lambda \vdash n$ that have no part equal to 1, so we have

$$\psi_{2m+1}^{(L)}(C_\lambda) = -(2m+1)! \omega_{2m}^*(C_\lambda) \quad \text{for all } \lambda \vdash n,$$

with $\epsilon = -1$, as asserted.

The degree of $\rho_n^{(L)}$ is $\psi_n^{(L)}(C_{[1^n]}) = \frac{n!}{n-1} = n \cdot (n-2)!$. It contains a copy of the one-dimensional representation $(\chi_{\text{sgn}})^n$ on S_n , so is reducible for $n \geq 2$.

(2) We have from (1) by transitivity of induction

$$\rho_{n-1}^+ := \text{Ind}_{S_{n-1}}^{S_n} (\text{Ind}_{C_{n-1}}^{S_{n-1}} ((\chi_{\text{sgn}})^n)).$$

We now identify the inner sum representation with ρ_{n-1}^ϵ , treating the cases n even and odd separately. For the case $n = 2m$, we use Theorem 5.1 to find that the inner induced representation on the right is the genuine representation

$$\rho_{n-1}^- = \text{Ind}_{C_{n-1}}^{S_{n-1}} (\chi_{\text{triv}}) = \text{Ind}_{C_{2m-1}}^{S_{2m-1}} (\chi_{\text{triv}}),$$

having character $(2m-1)! \omega_{2m-1}$. For the case $n = 2m+1$ we use Theorem 5.2 to find that the inner induced representation on the right is the genuine representation

$$\rho_n^+ = \text{Ind}_{C_{n-1}}^{S_{n-1}} (\chi_{\text{sgn}}),$$

having character $-(2m)! \omega_{2m}$, which completes (2). \square

5.5. The rescaled 1-splitting measure $n! \nu_{1,n}$ is the character of a virtual representation of S_n . The results above imply that the 1-splitting measure $\nu_{n,1}^*$ scaled by a factor $n!$ is the character of a virtual representation of S_n .

Theorem 5.5. *The class function $(-1)^{n-1} n! \nu_{n,1}$ of the scaled 1-splitting measure is the character of a virtual representation $\rho_{n,1}$ of S_n .*

(1) For even $n = 2m$, we have

$$\rho_{2m,1} = (\rho_{2m}^-)^{-1} \oplus \rho_{2m-1}^L.$$

(2) For odd $n = 2m+1$ we have

$$\rho_{2m+1,1} = \rho_{2m+1}^+ \oplus (\rho_{2m}^L)^{-1}.$$

Proof. On the character level we have

$$n! \nu_{n,1} = n! \omega_n + n! \omega_{n-1}^*.$$

It follows from Theorems 5.1, 5.2 and 5.4, that exactly one of the two terms on the right is the character of a genuine representation and the other the character of the

negative of a genuine representation. The answers depend on the parity of n , as given. \square

6. CONCLUDING REMARKS

It remains to put these results in a suitable geometric context, and identifying their possible relationship to the theory of \mathbb{F}_1 -objects.

- (1) As mentioned in the introduction, the elements of S_n having conjugacy classes with the partitions of the shapes in Theorem 4.1 (1) are exactly the *regular elements* of the Coxeter group S_n , in the sense of Springer [25, Sec. 5.1]. We showed this via the calculation in Lemma 2.5. It would be of interest to give a “geometric” explanation of the appearance of the Springer regular elements in the limiting 1-splitting distribution.
- (2) The Springer regular elements play a role in the “cyclic sieving phenomenon” of Reiner, Stanton and White [21]. See also the work of Reiner, Stanton and Webb [20] and the survey of Sagan [23].
- (3) It might be of interest to study further beyond Section 5.3 the multiplicities of irreducible representations of the decomposition of the induced representations of S_n into irreducible representations, in relation to the multiplicative structure of integers. The occurrence of signs in the constructions of this paper might be interpretable in terms of a “supersymmetry” operation.

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